Empirical Orthogonal Function Analysis of the Internal Wave Equation in Basins

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Background & Introduction

Basin-scale internal waves are an important driver of transport within lakes. Our goals are to model basin-scale waves within lakes, such as Crater Lake in Figure 1, and to compare analytically obtained solutions with those obtained from data-driven methods.

Figure 1: A photograph of Crater Lake Oregon, USA, taken by an astronaut member of the Expedition 52 crew aboard the ISS [1].

A model for the forced, linear, inviscid internal wave equation with the Boussinesq approximation for the vertical velocity, w, is given by:

$$\partial_{tt}\nabla^2 w + N^2 \nabla_\perp^2 w = \tilde{F}$$

Where $\nabla_\perp^2$ is the horizontal component of the Laplacian and $\tilde{F} := \partial_t \nabla_\perp^2 F$ where F acts on the vertical momentum. In addition to this, we imposed radial-symmetry of our solutions. Importantly, this model is non-homogeneous & accounts for non-uniform stratification.

We solved the equation in cylindrical coordinates using separation of variables to obtain the eigenfunctions.

Notably, our analytical eigenfunctions did **not** maintain orthogonality between the vertical and radial modes. As seen in Figure 2, even when resonantly forcing our 1,1-mode, we get non-zero inner products in mixed modes.

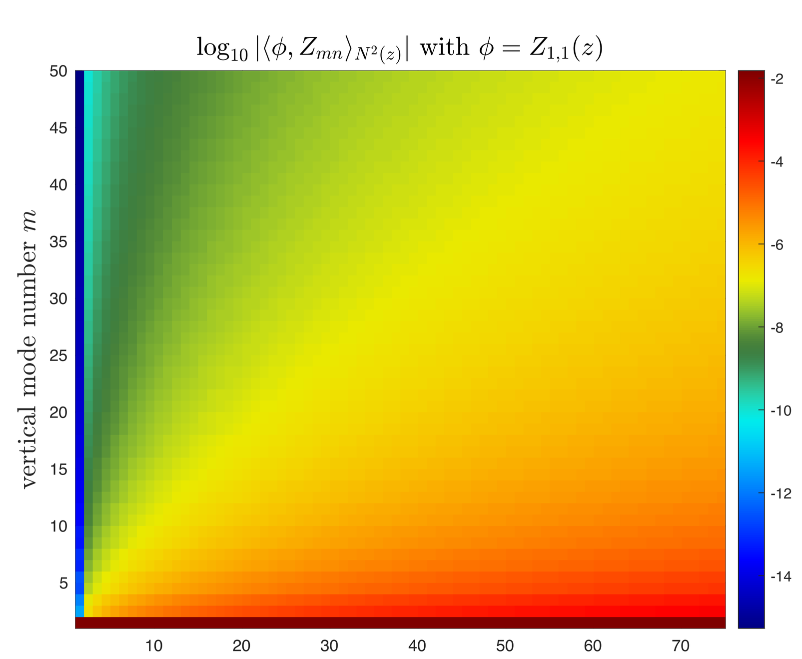
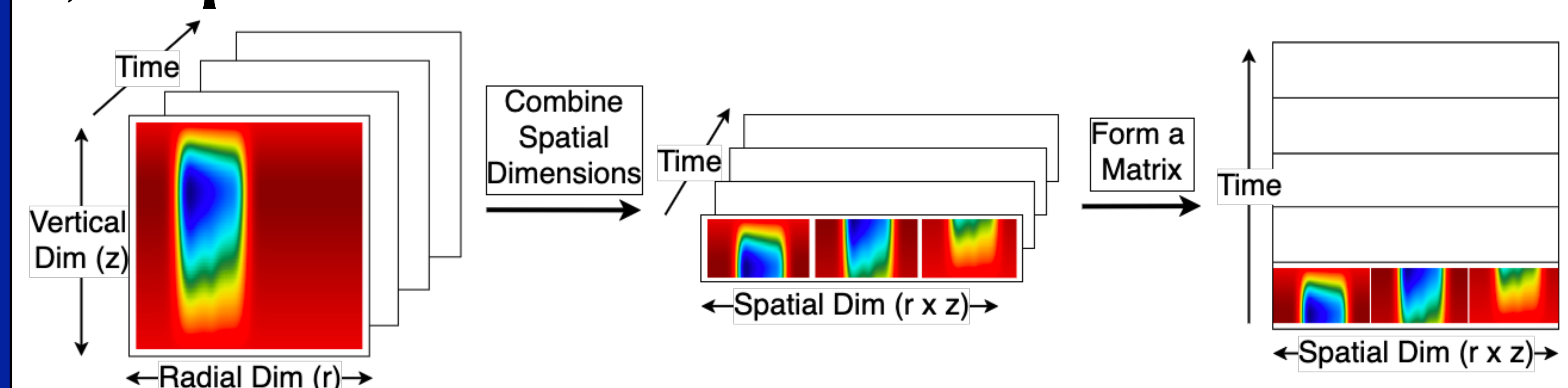


Figure 2: Log plot of the weighted inner product of the 1,1-mode with the n,m-modes.

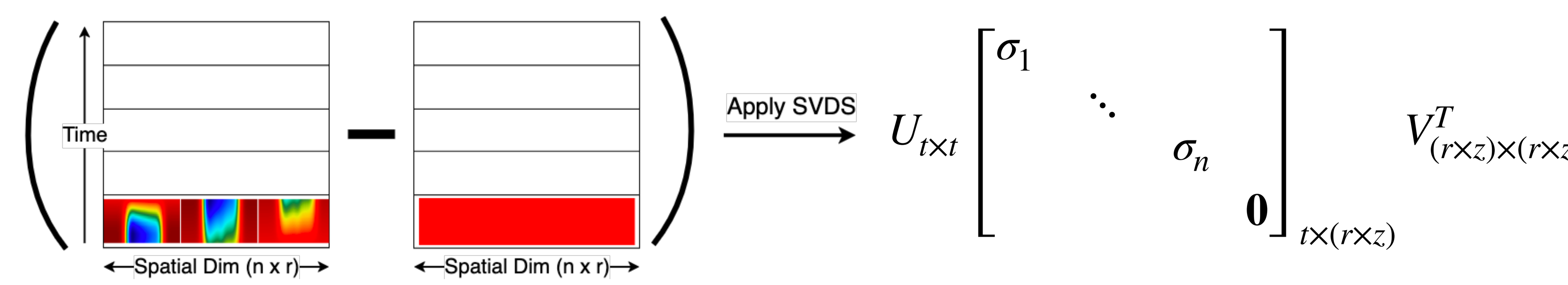
The lack of orthogonality between vertical and radial modes motivated us to apply Empirical Orthogonal Function (EOF) analysis - a data-driven dimension reduction technique that constructs an optimal orthogonal basis which captures the most variance of the data [2]. We then wanted to compare the performance of the EOFs to that of the analytical solutions obtained by our separation of variables.

Methods & Analysis

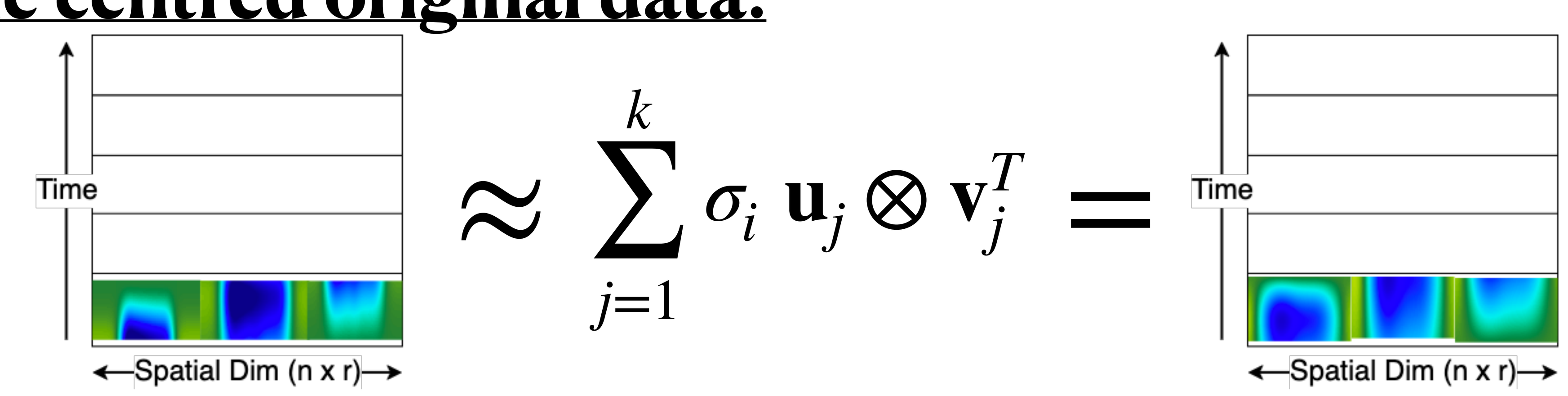
1) Preprocess data:



2) Centre data & apply SVDS:



3) Use a subset of the singular values to approximate the centred original data:



4) Evaluate the accuracy of the approximation:

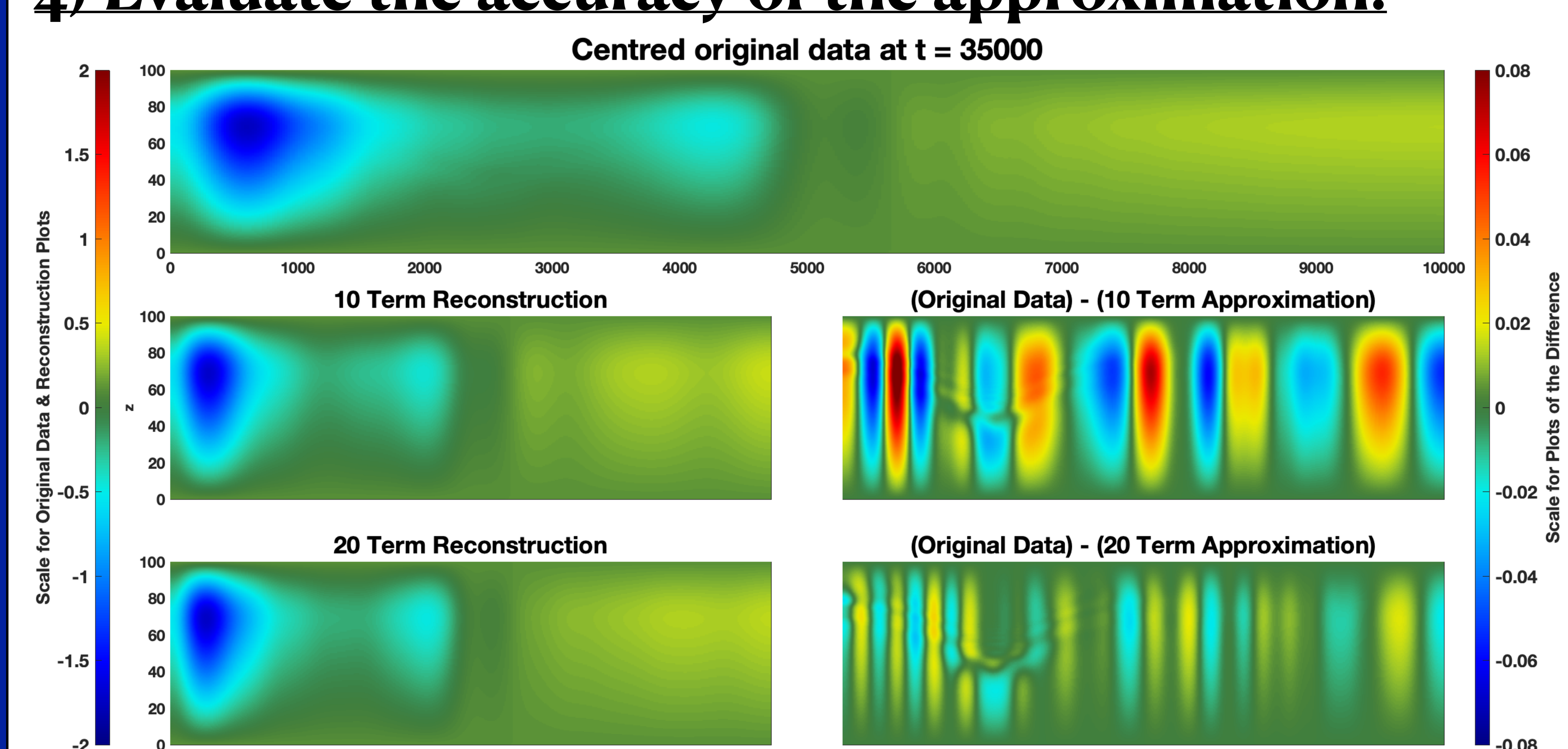


Figure 3: The analytic solution (top panel), the 10 and 20 term EOF approximations (left panels), as well as the difference between the analytic solution and the approximations (right panels) at time $t = 35000$.

Conclusions & Future Work

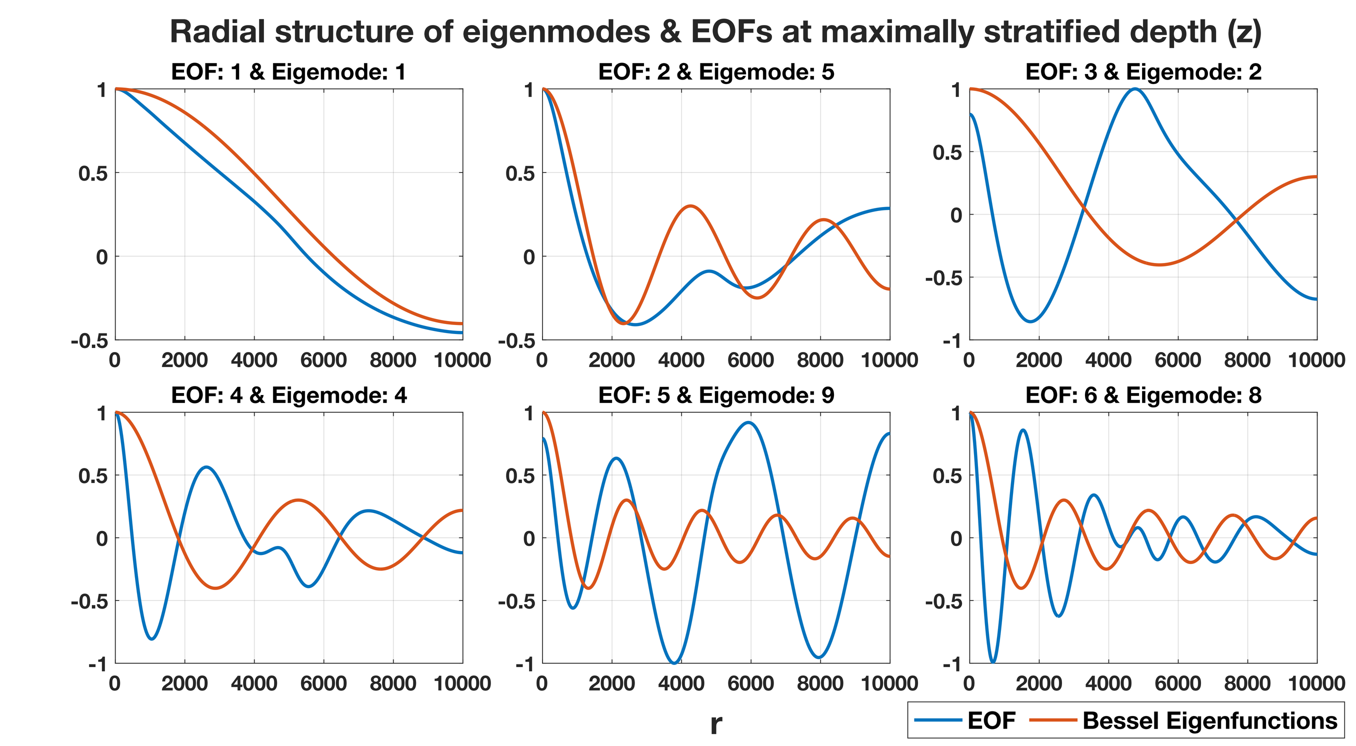


Figure 4: Plot of the radial structure of the first 6 EOFs (in blue) as well as the 6 largest eigenfunctions from the separation of variables (in orange). Notably, we see that our fifth EOF has consistently large oscillations which is not the case for the analytic eigenfunctions.

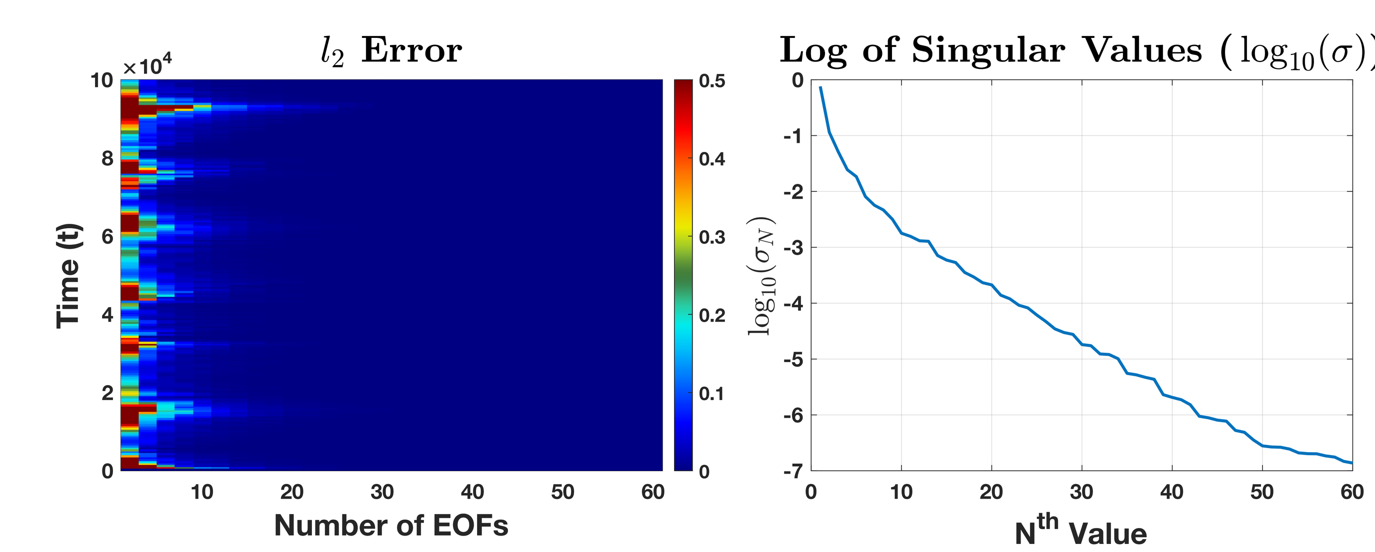


Figure 5: Plot of the l_2 error at each time for a varying number of EOFs (left) and a log plot of the first 60 singular values, σ (right). This figure demonstrates the effectiveness of the EOFs to reconstruct the original data with relatively few terms and shows that the magnitude of the EOFs decreases rapidly.

Moving forward, we aim to apply EOF analysis to field data to model and analyze basin-scale internal waves within real-world natural environments. Additionally, we aim to apply our analysis to non-linear equations such as the BBM equation or the Kuramoto–Sivashinsky equation.

References

[1] "Crater Lake.". [Online]. Available: https://earthobservatory.nasa.gov/images/90647/crater-lake

[2] A. Navarra and V. Simoncini, *A Guide to Empirical Orthogonal Functions for Climate Data Analysis*. Dordrecht: Springer Netherlands, 2010. doi: 10.1007/978-90-481-3702-2.

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